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## ► To cite this version:

Da-Yan Liu, Olivier Gibaru, Wilfrid Perruquetti. Parameters estimation of a noisy sinusoidal signal with time-varying amplitude. 19th Mediterranean Conference on Control and Automation (2011), Jun 2011, Corfou, Greece. inria-00593408

**HAL Id: inria-00593408**

**<https://inria.hal.science/inria-00593408>**

Submitted on 16 May 2011

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# Parameters estimation of a noisy sinusoidal signal with time-varying amplitude

Da-yan Liu, Olivier Gibaru and Wilfrid Perruquetti

**Abstract**—In this paper, we give estimators of the frequency, amplitude and phase of a noisy sinusoidal signal with time-varying amplitude by using the algebraic parametric techniques introduced by Fliess and Sira-Ramírez. We apply a similar strategy to estimate these parameters by using modulating functions method. The convergence of the noise error part due to a large class of noises is studied to show the robustness and the stability of these methods. We also show that the estimators obtained by modulating functions method are robust to “large” sampling period and to non zero-mean noises.

## I. INTRODUCTION

Recent algebraic parametric estimation techniques for linear systems [1], [2], [3] have been extended to various problems in signal processing (see, e.g., [4], [5], [6], [7], [8]). In [9], [10], [11], these methods are devoted to estimate the frequency, amplitude and phase of a noisy sinusoidal signal with time-invariant amplitude. Let us emphasize that these methods, which are algebraic and non-asymptotic, exhibit good robustness properties with respect to corrupting noises, without the need of knowing their statistical properties (see [12], [13] for more theoretical details). We have shown in [14] that the differentiation estimators proposed by algebraic parametric techniques can cope with a large class of noises for which the mean and covariance are polynomials in time. The robustness properties have already been confirmed by numerous computer simulations and several laboratory experiments. In [15], [9], modulating functions methods are used to estimate unknown parameters of noisy sinusoidal signals. These methods have similar advantages than algebraic parametric techniques especially concerning the robustness of estimations to corrupting noises. The aim of this paper is to estimate the frequency, amplitude and phase of a noisy time-varying amplitude sinusoidal signal by using the previous two methods. We also show their stability by studying the convergence of the noise error part due to a large class of noises.

In Section II, we give some notations and useful formulae. In Section III and Section IV, we give parameters’ estimators by using respectively algebraic parametric techniques and

modulating functions method. In Section V, the estimators are given in discrete case. Then, we study the influence of sampling period on the associated noise error part due to a class of noises. In Section VI, inspired by [15] a recursive algorithm for the frequency estimator is given, then some numerical simulations are given to show the efficiency and stability of our estimators.

## II. NOTATIONS PRELIMINARIES

Let us denote by  $D_T := \{T \in \mathbb{R}_+^* : [0, T] \subset \Omega\}$ , and  $w_{\mu, \kappa}(\tau) = (1 - \tau)^\mu \tau^\kappa$  for any  $\tau \in [0, 1]$  with  $\mu, \kappa \in ]-1, +\infty[$ . By using the Rodrigues formula (see [16] p.67), we have

$$\frac{d^i}{d\tau^i} \{w_{\mu, \kappa}(\tau)\} = (-1)^i i! w_{\mu-i, \kappa-i}(\tau) P_i^{\mu-i, \kappa-i}(\tau), \quad (1)$$

where  $P_i^{\mu-i, \kappa-i}$ ,  $\min(\kappa, \mu) \geq i \in \mathbb{N}$ , is the  $i^{\text{th}}$  order Jacobi polynomial defined on  $[0, 1]$  (see [16]):  $\forall \tau \in [0, 1]$ ,

$$P_i^{\mu-i, \kappa-i}(\tau) = \sum_{s=0}^i (-1)^{i-s} \binom{\mu}{s} \binom{\kappa}{i-s} w_{i-s, i}(\tau). \quad (2)$$

Then, we have the following lemma.

**Lemma 1:** Let  $f$  be a  $\mathcal{C}^{n+1}(\Omega)$ -continuous function ( $n \in \mathbb{N}$ ) and  $\Pi_{k, \mu}^n$  be a differential operator defined as follows

$$\Pi_{k, \mu}^n = \frac{1}{s^{n+1+\mu}} \cdot \frac{d^{n+k}}{ds^{n+k}} \cdot s^n, \quad (3)$$

where  $s$  is the Laplace variable,  $k \in \mathbb{N}$  and  $-1 < \mu \in \mathbb{R}$ . Then, the inverse Laplace transform of  $\Pi_{k, \mu}^n \hat{f}$  where  $\hat{f}$  is the laplace transformation of  $f$  is given by

$$\begin{aligned} & \mathcal{L}^{-1} \{ \Pi_{k, \mu}^n \hat{f}(s) \} (T) \\ &= T^{n+1+\mu+k} c_{\mu+n, k} \int_0^1 w_{\mu+n, k+n}^{(n)}(\tau) f(T\tau) d\tau, \end{aligned} \quad (4)$$

where  $T \in D_T$  and  $c_{\mu+n, k} = \frac{(-1)^k}{\Gamma(\mu+n+1)}$ .

In order to prove this lemma, let us recall that the  $\alpha$ -order ( $\alpha \in \mathbb{R}^+$ ) Riemann-Liouville integral (see [17]) of a real function  $g : (\mathbb{R} \rightarrow \mathbb{R})$  is defined by

$$J^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} g(\tau) d\tau. \quad (5)$$

The associated Laplace transform is given by

$$\mathcal{L} \{ J^\alpha g(t) \} (s) = s^{-\alpha} \hat{g}(s), \quad (6)$$

where  $\hat{g}$  denotes the Laplace transform of  $g$ .

**Proof.** Let us denote  $W_{\mu+n, \kappa+n}(t) = (T - t)^{\mu+n} t^{\kappa+n}$  for any  $t \in [0, T]$ . Then, by applying the Laplace transform to

D.Y. Liu is with Paul Painlevé (CNRS, UMR 8524), Université de Lille 1, 59650, Villeneuve d’Ascq, France [dayan.liu@inria.fr](mailto:dayan.liu@inria.fr)

O. Gibaru is with Laboratory of Applied Mathematics and Metrology (L2MA), Arts et Metiers ParisTech, 8 Boulevard Louis XIV, 59046 Lille Cedex, France [olivier.gibaru@ensam.eu](mailto:olivier.gibaru@ensam.eu)

W. Perruquetti is with LAGIS (CNRS, UMR 8146), École Centrale de Lille, BP 48, Cité Scientifique, 59650 Villeneuve d’Ascq, France [wilfrid.perruquetti@inria.fr](mailto:wilfrid.perruquetti@inria.fr)

D.Y. Liu, O. Gibaru and W. Perruquetti are with Équipe Projet Non-A, INRIA Lille-Nord Europe, Parc Scientifique de la Haute Borne 40, avenue Halley Bât.A, Park Plaza, 59650 Villeneuve d’Ascq, France

the following Riemann-Liouville integral and doing some classical operational calculations, we obtain

$$\begin{aligned} & \mathcal{L} \left\{ c_{\mu+n,k+n} \int_0^T W_{\mu+n,k+n}(\tau) f^{(n)}(\tau) d\tau \right\} \\ &= s^{-(n+1+\mu)} \mathcal{L} \left\{ (-1)^{n+k} \tau^{n+k} f^{(n)}(\tau) \right\} \\ &= s^{-(n+1+\mu)} \frac{d^{n+k}}{ds^{n+k}} \mathcal{L} \left\{ f^{(n)}(\tau) \right\} \\ &= s^{-(n+1+\mu)} \frac{d^{n+k}}{ds^{n+k}} s^n \hat{f}(s) = \Pi_{k,\mu}^n \hat{f}(s). \end{aligned}$$

Then, by substituting  $\tau$  by  $T\tau$  we have

$$\begin{aligned} & c_{\mu+n,k+n} \int_0^T W_{\mu+n,k+n}(\tau) f^{(n)}(\tau) d\tau \\ &= T^{2n+k+\mu+1} c_{\mu+n,k+n} \int_0^1 w_{\mu+n,k+n}(\tau) f^{(n)}(T\tau) d\tau. \end{aligned} \quad (7)$$

By using (1), we obtain  $w_{\mu+n,k+n}^{(i)}(0) = w_{\mu+n,k+n}^{(i)}(1) = 0$  for  $i = 0, \dots, n-1$ . Finally, this proof can be completed by applying  $n$  times integration by parts to (7).  $\square$

### III. ALGEBRAIC PARAMETRIC TECHNIQUES

Let  $y = x + \bar{w}$  be a noisy observation on a finite time interval  $\Omega \subset \mathbb{R}^+$  of a real valued signal  $x$ , where  $\bar{w}$  is an additive corrupting noise and

$$\forall t \in \Omega, x(t) = (A_0 + A_1 t) \sin(\omega t + \phi) \quad (8)$$

with  $A_0 \in \mathbb{R}_+^*$ ,  $A_1 \in \mathbb{R}^*$ ,  $\omega \in \mathbb{R}_+^*$  and  $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$ . Observe that  $x$  is a time-variant varying sinusoidal signal, which is a solution to the harmonic oscillator equation

$$\forall t \in \Omega, x^{(4)}(t) + 2\omega^2 \ddot{x}(t) + \omega^4 x(t) = 0. \quad (9)$$

Then, we can estimate the parameters  $\omega$ ,  $A_0$  and  $\phi$  by applying algebraic parametric techniques to (9).

**Proposition 1:** Let  $k \in \mathbb{N}$ ,  $-1 < \mu \in \mathbb{R}$  and  $T \in D_T$  such that  $A_1 \int_0^1 \dot{w}_{\mu+4,k+4}(\tau) \sin(\omega T\tau + \phi) d\tau \leq 0$ , then the parameter  $\omega$  is estimated from the noisy observation  $y$  by

$$\tilde{\omega} = \left( \frac{-B_y + \sqrt{B_y^2 - 4A_y C_y}}{2A_y} \right)^{\frac{1}{2}}, \quad (10)$$

where  $A_y = T^4 \int_0^1 w_{\mu+4,k+4}(\tau) y(T\tau) d\tau$ ,  $B_y = 2T^2 \int_0^1 \dot{w}_{\mu+4,k+4}(\tau) y(T\tau) d\tau$ ,  $C_y = \int_0^1 w_{\mu+4,k+4}^{(4)}(\tau) y(T\tau) d\tau$ ,  $w_{\mu+4,k+4}^{(i)}$  is given by (1) with  $i = 1, 2, 4$ .

**Proof.** By applying the Laplace transform to (9), we get

$$\begin{aligned} & s^4 \hat{x}(s) + 2\omega^2 s^2 \hat{x}(s) + \omega^4 \hat{x}(s) \\ &= s^3 x_0 + s^2 \dot{x}_0 + (2\omega^2 x_0 + \ddot{x}_0)s + (2\omega^2 \dot{x}_0 + x_0^{(3)}). \end{aligned} \quad (11)$$

Let us apply  $k+4$  ( $k \in \mathbb{N}$ ) times derivations to both sides of (11) with respect to  $s$ . By multiplying the resulting equation by  $s^{-5-\mu}$  with  $-1 < \mu \in \mathbb{R}$ , we get

$$\Pi_{k,\mu}^4 \hat{x}(s) + 2\omega^2 \Pi_{k+2,\mu+2}^2 \hat{x}(s) + \omega^4 \Pi_{k+4,\mu+4}^0 \hat{x}(s) = 0. \quad (12)$$

Let us apply the inverse Laplace transform to (12), then by using Lemma 1, we obtain

$$\begin{aligned} & \int_0^1 \left( w_{\mu+4,k+4}^{(4)}(\tau) + 2(\omega T)^2 \ddot{w}_{\mu+4,k+4}(\tau) \right) x(T\tau) d\tau \\ &+ (\omega T)^4 \int_0^1 w_{\mu+4,k+4}(\tau) x(T\tau) d\tau = 0. \end{aligned}$$

According to (1), we have  $w_{\mu+4,k+4}^{(i)}(0) = w_{\mu+4,k+4}^{(i)}(1)$  for  $i = 0, \dots, 3$ . Then by applying integration by parts, we get

$$\begin{aligned} & \omega^4 \int_0^1 w_{\mu+4,k+4}(\tau) x(T\tau) d\tau + 2\omega^2 \int_0^1 \ddot{w}_{\mu+4,k+4}(\tau) x^{(2)}(T\tau) d\tau \\ &+ \int_0^1 w_{\mu+4,k+4}(\tau) x^{(4)}(T\tau) d\tau = 0. \end{aligned}$$

Thus,  $\omega^2$  is obtained by

$$\omega^2 = \frac{-\hat{B}_x \pm \sqrt{\hat{B}_x^2 - 4\hat{A}_x \hat{C}_x}}{2\hat{A}_x}, \quad (13)$$

where  $\hat{A}_x = \int_0^1 w_{\mu+4,k+4}(\tau) x(T\tau) d\tau$ ,  $\hat{B}_x = 2 \int_0^1 \ddot{w}_{\mu+4,k+4}(\tau) x^{(2)}(T\tau) d\tau$ ,  $\hat{C}_x = \int_0^1 w_{\mu+4,k+4}(\tau) x^{(4)}(T\tau) d\tau$ . Since  $x^{(4)}(T\tau) + 2\omega^2 x^{(2)}(T\tau) + \omega^4 x(T\tau) = 0$  for any  $\tau \in [0, 1]$ , we get

$$\begin{aligned} & \frac{1}{4} (\hat{B}_x^2 - 4\hat{A}_x \hat{C}_x) = \\ & \left( \int_0^1 w_{\mu+4,k+4}(\tau) x^{(2)}(T\tau) d\tau + \int_0^1 \omega^2 w_{\mu+4,k+4}(\tau) x(T\tau) d\tau \right)^2. \end{aligned}$$

Observe that  $x^{(2)}(T\tau) + \omega^2 x(T\tau) = 2\omega A_1 \cos(\omega Tt + \phi)$  for any  $\tau \in [0, 1]$ . If  $\omega A_1 \int_0^1 w_{\mu+4,k+4}(\tau) \cos(\omega Tt + \phi) d\tau = -\frac{A_1}{T} \int_0^1 \dot{w}_{\mu+4,k+4}(\tau) \sin(\omega T\tau + \phi) d\tau \geq 0$ , then we get

$$\frac{-\hat{B}_x + \sqrt{\hat{B}_x^2 - 4\hat{A}_x \hat{C}_x}}{2\hat{A}_x} = \omega^2. \quad (14)$$

Finally, this proof can be completed by applying integration by parts and substituting  $x$  by  $y$  in the last equation.  $\square$

By observing that  $x_0 = x(0) = A_0 \sin \phi$ ,  $\dot{x}_0 = \dot{x}(0) = A_0 \omega \cos \phi + A_1 \sin \phi$  and  $x_0^{(3)} = x^{(3)}(0) = -\omega^2 \dot{x}_0 - 2\omega^2 A_1 \sin \phi$ , then we can obtain  $A_0 \cos \phi = \frac{1}{2\omega^3} (x_0^{(3)} + 3\omega^2 \dot{x}_0)$ . Hence, if  $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ , then we have

$$\begin{aligned} & A_0 = \left( x_0^2 + \frac{(x_0^{(3)} + 3\omega^2 \dot{x}_0)^2}{4\omega^6} \right)^{\frac{1}{2}}, \\ & \phi = \arctan \left( \frac{2\omega^3 x_0}{x_0^{(3)} + 3\omega^2 \dot{x}_0} \right). \end{aligned} \quad (15)$$

Thus, we need to estimate  $x_0$ ,  $\dot{x}_0$  and  $x_0^{(3)}$  so as to obtain the estimations of  $A_0$  and  $\phi$ .

**Proposition 2:** Let  $-1 < \mu \in \mathbb{R}$  and  $T \in D_T$ , then the parameters  $A_0$  and  $\phi$  are estimated from the noisy observation

y and the estimated value of  $\omega$  given in (10):

$$\begin{aligned}\tilde{A}_0 &= \left( \tilde{x}_0^2 + \frac{(\tilde{x}_0^{(3)} + 3\tilde{\omega}^2 \tilde{x}_0)^2}{4\tilde{\omega}^6} \right)^{\frac{1}{2}}, \\ \tilde{\phi} &= \arctan \left( \frac{2\tilde{\omega}^3 \tilde{x}_0}{\tilde{x}_0^{(3)} + 3\tilde{\omega}^2 \tilde{x}_0} \right),\end{aligned}\quad (16)$$

where

$$\begin{aligned}\tilde{x}_0 &= \int_0^1 P_2^{\tilde{\omega}}(\tau) y(T\tau) d\tau, \quad \tilde{x}_0 = \frac{1}{T} \int_0^1 P_3^{\tilde{\omega}}(\tau) y(T\tau) d\tau, \\ \tilde{x}_0^{(3)} &= \frac{1}{T^3} \int_0^1 P_4^{\tilde{\omega}}(\tau) y(T\tau) d\tau - 2\tilde{\omega}^2 \tilde{x}_0, \\ \frac{6}{\Gamma(\mu+5)} P_2^{\tilde{\omega}}(\tau) &= \sum_{i=0}^3 \binom{3}{i} \frac{4! c_{\mu+i,3-i}}{(4-i)!} w_{\mu+i,3-i}(\tau) \\ &+ 4(\tilde{\omega}T)^2 \sum_{i=0}^2 \binom{3}{i} \frac{c_{\mu+i+2,3-i}}{(2-i)!} w_{\mu+i+2,3-i}(\tau) \\ &+ (\tilde{\omega}T)^4 c_{\mu+4,3} w_{\mu+4,3}(\tau), \\ \frac{-2}{\Gamma(\mu+6)} P_3^{\tilde{\omega}}(\tau) &= c_{\mu,3} w_{\mu,3}(\tau) + 11c_{\mu+1,2} w_{\mu+1,2}(\tau) \\ &+ 28c_{\mu+2,1} w_{\mu+2,1}(\tau) + 12c_{\mu+3,0} w_{\mu+3,0}(\tau) \\ &+ 2(\tilde{\omega}T)^2 (c_{\mu+2,3} w_{\mu+2,3}(\tau) + 5c_{\mu+3,2} w_{\mu+3,2}(\tau)) \\ &+ 4(\tilde{\omega}T)^2 (c_{\mu+4,1} w_{\mu+4,1}(\tau) - c_{\mu+5,0} w_{\mu+5,0}(\tau)) \\ &+ (\tilde{\omega}T)^4 (c_{\mu+4,3} w_{\mu+4,3}(\tau) - c_{\mu+5,2} w_{\mu+5,2}(\tau)), \\ \frac{-6}{\Gamma(\mu+8)} P_4^{\tilde{\omega}}(\tau) &= \sum_{i=0}^3 \binom{3}{i} \frac{3! c_{\mu+i,3-i}}{(3-i)!} w_{\mu+i,3-i}(\tau) \\ &+ 2(\tilde{\omega}T)^2 \sum_{i=0}^1 \binom{3}{i} c_{\mu+i+2,3-i} w_{\mu+i+2,3-i}(\tau) \\ &+ (\tilde{\omega}T)^4 \sum_{i=0}^3 \binom{3}{i} (-1)^i i! c_{\mu+4+i,3-i} w_{\mu+4+i,3-i}(\tau).\end{aligned}$$

**Proof.** In order to estimate  $x_0$ , we apply the following operator  $\Pi_1 = \frac{1}{s^{\mu+5}} \cdot \frac{d^3}{ds^3}$  to (11) with  $-1 < \mu \in \mathbb{R}$ , which annihilates each terms containing  $x_0^{(i)}$  for  $i = 1, 2, 3$ . Then, by using the Leibniz formula, we get

$$\begin{aligned}\frac{6}{s^{\mu+5}} x_0 &= \sum_{i=0}^3 \binom{3}{i} \frac{4!}{(4-i)!} \frac{1}{s^{\mu+1+i}} \hat{x}^{(3-i)}(s) \\ &+ 2\omega^2 \sum_{i=0}^2 \binom{3}{i} \frac{2!}{(2-i)!} \frac{1}{s^{\mu+3+i}} \hat{x}^{(3-i)}(s) + \frac{\omega^4}{s^{\mu+5}} \hat{x}^{(3)}(s).\end{aligned}$$

Let us express the last equation in the time domain. By denoting  $T$  as the length of the estimation time window we have

$$\begin{aligned}\frac{6T^{\mu+4}}{\Gamma(\mu+5)} x_0 &= \int_0^T \sum_{i=0}^3 \binom{3}{i} \frac{4! c_{\mu+i,3-i}}{(4-i)!} w_{\mu+i,3-i}(\tau) d\tau \\ &+ 4\omega^2 \int_0^T \sum_{i=0}^2 \binom{3}{i} \frac{c_{\mu+i+2,3-i}}{(2-i)!} w_{\mu+i+2,3-i}(\tau) d\tau \\ &+ \omega^4 \int_0^T c_{\mu+4,3} w_{\mu+4,3}(\tau) d\tau.\end{aligned}$$

Hence, by substituting  $\tau$  by  $T\tau$ ,  $x$  by  $y$  and taking the estimation of  $\omega$  given in Proposition 1 we obtain an estimate for  $x_0$ . Similarly, we apply the operator  $\Pi_2 = \frac{1}{s^{\mu+4}} \cdot \frac{d}{ds} \cdot \frac{1}{s} \cdot \frac{d^2}{ds^2}$  (resp.  $\Pi_3 = \frac{1}{s^{\mu+4}} \cdot \frac{d^3}{ds^3} \cdot \frac{1}{s}$ ) to (11) to compute an estimate for  $\dot{x}_0$  (resp.  $x_0^{(3)}$ ). Finally, we get estimations for  $A_0$  and  $\phi$  from relations (15) by using the estimations of  $x_0$ ,  $\dot{x}_0$ ,  $x_0^{(3)}$  and  $\omega$ .  $\square$

#### IV. MODULATING FUNCTIONS METHOD

**Proposition 3:** Let  $f$  be a function belonging to  $\mathcal{C}^4([0, 1])$  which satisfies the following conditions  $f^{(i)}(0) = f^{(i)}(1)$  for  $i = 0, \dots, 3$ . Assume that  $A_1 \int_0^1 \dot{f}(\tau) \sin(\omega T\tau + \phi) d\tau \leq 0$  with  $T \in D_T$ , then the parameter  $\omega$  is estimated from the noisy observation  $y$  by

$$\tilde{\omega} = \left( \frac{-B_y + \sqrt{B_y^2 - 4A_y C_y}}{2A_y} \right)^{\frac{1}{2}}, \quad (17)$$

where  $A_y = T^4 \int_0^1 f(\tau) y(T\tau) d\tau$ ,  $B_y = 2T^2 \int_0^1 \dot{f}(\tau) y(T\tau) d\tau$ ,  $C_y = \int_0^1 f^{(4)}(\tau) y(T\tau) d\tau$ .

**Proof.** Recall that  $x^{(4)}(T\tau) + 2\omega^2 x^{(2)}(T\tau) + \omega^4 x(T\tau) = 0$  for any  $\tau \in [0, 1]$ . As  $f$  is continuous on  $[0, 1]$ , then we have

$$\begin{aligned}\int_0^1 f(\tau) x^{(4)}(T\tau) d\tau + 2\omega^2 \int_0^1 f(\tau) x^{(2)}(T\tau) d\tau \\ + \omega^4 \int_0^1 f(\tau) x(T\tau) d\tau = 0.\end{aligned}$$

Then, this proof can be completed similarly to the one of Proposition 1.  $\square$

**Proposition 4:** Let  $f_i$  for  $i = 1, \dots, 4$  be four continuous functions defined on  $[0, 1]$ . Assume that there exists  $T \in D_T$  such that the determinant of the matrix  $M_\omega = (M_{i,j}^\omega)_{1 \leq i, j \leq 4}$  is different to zero, where for  $i = 1, \dots, 4$

$$\begin{aligned}M_{i,1}^\omega &= \int_0^1 f_i(\tau) \sin(\omega T\tau) d\tau, M_{i,3}^\omega = \int_0^1 f_i(\tau) T\tau \sin(\omega T\tau) d\tau, \\ M_{i,2}^\omega &= \int_0^1 f_i(\tau) \cos(\omega T\tau) d\tau, M_{i,4}^\omega = \int_0^1 f_i(\tau) T\tau \cos(\omega T\tau) d\tau.\end{aligned}$$

Then, for any  $\phi \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$  the estimations of  $A_0$ ,  $A_1$  and  $\phi$  are given by

$$\begin{aligned}\tilde{A}_i &= \left( (A_i \tilde{\cos} \phi)^2 + (A_i \tilde{\sin} \phi)^2 \right)^{1/2}, \\ \tilde{\phi} &= \arctan \left( \frac{A_0 \tilde{\sin} \phi}{A_0 \tilde{\cos} \phi} \right),\end{aligned}\quad (18)$$

where the estimates of  $A_i \cos \phi$  and  $A_i \sin \phi$  for  $i = 0, 1$  are obtained by solving the following linear system

$$M_{\tilde{\omega}} \begin{pmatrix} A_0 \tilde{\cos} \phi \\ A_0 \tilde{\sin} \phi \\ A_1 \tilde{\cos} \phi \\ A_1 \tilde{\sin} \phi \end{pmatrix} = \begin{pmatrix} I_{f_1}^y \\ I_{f_2}^y \\ I_{f_3}^y \\ I_{f_4}^y \end{pmatrix}, \quad (19)$$

where  $I_{f_i}^y = \int_0^1 f_i(\tau) y(T\tau) d\tau$  for  $i = 1, \dots, 4$ , and  $\tilde{\omega}$  is the estimate of  $\omega$  given by Proposition 3.

**Proof.** Let us take an expansion of  $x$

$$x(T\tau) = A_0 \cos \phi \sin(\omega T\tau) + A_0 \sin \phi \cos(\omega T\tau) \\ + A_1 \cos \phi T\tau \sin(\omega T\tau) + A_1 \sin \phi T\tau \cos(\omega T\tau),$$

where  $\tau \in [0, 1]$ ,  $T \in D_T$ . By multiplying both sides of the last equation by the continuous functions  $f_i$  for  $i = 1, \dots, 4$  and by integrating the resulting equations between 0 and 1, we obtain

$$I_{f_i}^x = A_0 \cos \phi M_{i,1}^\omega + A_0 \sin \phi M_{i,2}^\omega + A_1 \cos \phi M_{i,3}^\omega + A_1 \sin \phi M_{i,4}^\omega.$$

Then, it yields the following linear system

$$M_\omega \begin{pmatrix} A_0 \cos \phi \\ A_0 \sin \phi \\ A_1 \cos \phi \\ A_1 \sin \phi \end{pmatrix} = \begin{pmatrix} I_{f_1}^x \\ I_{f_2}^x \\ I_{f_3}^x \\ I_{f_4}^x \end{pmatrix}.$$

Since  $\det(M_\omega) \neq 0$ , we obtain  $A_i \cos \phi$  and  $A_i \sin \phi$  for  $i = 0, 1$ . Finally, the proof can be completed by substituting  $x$  by  $y$  in the so obtained formulae of  $A_i \cos \phi$  and  $A_i \sin \phi$ .  $\square$

From now on, we choose functions  $w_{\mu+n, \kappa+n}^{(n)}$  with  $n \in \mathbb{N}$ ,  $\mu, \kappa \in ]-1, +\infty[$  for the previous modulating functions. Consequently, the estimate for  $\omega$  given in Proposition 3 generalizes the estimate given in Proposition 1.

## V. ANALYSIS OF THE ERRORS DUE TO THE NOISE AND THE SAMPLING PERIOD

### A. Two different sources of errors

Let us assume now that  $y(t_i) = x(t_i) + \bar{\omega}(t_i)$  ( $t_i \in \Omega$ ) is a noisy measurement of  $x$  in discrete case with an equidistant sampling period  $T_s$ . Since  $y$  is a discrete measurement, we apply the trapezoidal numerical integration method to approximate the integrals used in the previous estimators. Let  $\tau_i = \frac{i}{m}$  and  $a_i > 0$  for  $i = 0, \dots, m$  with  $m = \frac{T}{T_s} \in \mathbb{N}^*$  (except for  $a_0 \geq 0$  and  $a_m \geq 0$ ) be respectively the abscissas and the weights for a given numerical integration method. Weight  $a_0$  (resp.  $a_m$ ) is set to zero in order to avoid the infinite value at  $\tau = 0$  when  $-1 < \kappa < 0$  (resp.  $\tau = 1$  when  $-1 < \mu < 0$ ). Let us denote by  $q$  the functions obtained in the integrals of our estimators. Then, we denote by  $I_q^y := \int_0^1 q(\tau) y(T\tau) d\tau$ .

Hence,  $I_q^y$  is approximated by  $I_q^{y,m} := \sum_{i=0}^m \frac{a_i}{m} q(\tau_i) y(T\tau_i)$ . By writing  $y(t_i) = x(t_i) + \bar{\omega}(t_i)$ , we get  $I_q^{y,m} = I_q^{x,m} + e_q^{\bar{\omega},m}$ , where  $e_q^{\bar{\omega},m} = \sum_{i=0}^m \frac{a_i}{m} q(\tau_i) \bar{\omega}(T\tau_i)$ . Thus the integral  $I_q^y$  is corrupted by two sources of errors:

- the numerical error which comes from the numerical integration method,
- the noise error contributions  $e_q^{\bar{\omega},m}$ .

In the next subsection, we study the choice for the sampling period so as to reduce the noise error contributions.

### B. Analysis of the noise error for different stochastic processes

We assume in this section that the additive corruption noise  $\{\bar{\omega}(t_i), t_i \in \Omega\}$  is a continuous stochastic process satisfying the following conditions

- (C<sub>1</sub>) : for any  $s, t \geq 0$ ,  $s \neq t$ ,  $\bar{\omega}(s)$  and  $\bar{\omega}(t)$  are independent;
- (C<sub>2</sub>) : the mean value function of  $\{\bar{\omega}(\tau), \tau \geq 0\}$  belongs to  $\mathcal{L}(\Omega)$ ;
- (C<sub>3</sub>) : the variance function of  $\{\bar{\omega}(\tau), \tau \geq 0\}$  is bounded on  $\Omega$ .

Note that white Gaussian noise and Poisson noise satisfy these conditions. When the value of  $T$  is set, then  $T_s \rightarrow 0$  is equivalent to  $m \rightarrow +\infty$ . We are going to show the convergence of the noise error contributions when  $T_s \rightarrow 0$ .

**Lemma 2:** Let  $\bar{\omega}(t_i)$  be a sequence of  $\{\bar{\omega}(\tau), \tau \geq 0\}$  with an equidistant sampling period  $T_s$ , where  $\{\bar{\omega}(\tau), \tau \geq 0\}$  be a continuous stochastic process satisfying conditions (C<sub>1</sub>) – (C<sub>3</sub>). Assume that  $q \in \mathcal{L}^2([0, 1])$ , then we have

$$\lim_{m \rightarrow +\infty} E[e_q^{\bar{\omega},m}] = \int_0^1 q(\tau) E[\bar{\omega}(T\tau)] d\tau, \\ \lim_{m \rightarrow +\infty} \text{Var}[e_q^{\bar{\omega},m}] = 0. \quad (20)$$

**Proof.** Since  $\bar{\omega}(t_i)$  is a sequence of independent random variables (C<sub>1</sub>), then by using the properties of mean value and variance functions we have

$$E[e_q^{\bar{\omega},m}] = \frac{1}{m} \sum_{i=0}^m a_i q(\tau_i) E[\bar{\omega}(T\tau_i)], \\ \text{Var}[e_q^{\bar{\omega},m}] = \frac{1}{m^2} \sum_{i=0}^m a_i^2 q^2(\tau_i) \text{Var}[\bar{\omega}(T\tau_i)]. \quad (21)$$

According to (C<sub>3</sub>), the variance function of  $\bar{\omega}$  is bounded. Then we have

$$0 \leq \frac{1}{m^2} \sum_{i=0}^m a_i^2 q^2(\tau_i) |\text{Var}[\bar{\omega}(T\tau_i)]| \leq U \frac{a(m)}{m} \sum_{i=0}^m \frac{a_i}{m} q^2(\tau_i), \quad (22)$$

where  $a(m) = \max_{0 \leq i \leq m} a_i$  and  $U = \sup_{0 \leq \tau \leq 1} |\text{Var}[\bar{\omega}(T\tau)]| < +\infty$ .

Moreover, since  $q \in \mathcal{L}^2([0, 1])$  and the mean value function of  $\bar{\omega}$  is integrable (C<sub>2</sub>), then we have

$$\lim_{m \rightarrow +\infty} E[e_q^{\bar{\omega},m}] = \int_0^1 q(\tau) E[\bar{\omega}(T\tau)] d\tau, \\ \lim_{m \rightarrow +\infty} \sum_{i=0}^m \frac{a_i}{m} q^2(\tau_i) = \int_0^1 q^2(\tau) d\tau < +\infty. \quad (23)$$

As all  $a_i$  are bounded, we have  $U \frac{a(m)}{m} \sum_{i=0}^m \frac{a_i}{m} q^2(\tau_i) = 0$ . This proof is completed.  $\square$

**Theorem 1:** With the same conditions given in Lemma 2, we have the following convergence

$$e_q^{\bar{\omega},m} \xrightarrow{\mathcal{L}^2([0,1])} \int_0^1 q(\tau) E[\bar{\omega}(T\tau)] d\tau, \text{ when } T_s \rightarrow 0. \quad (24)$$

Moreover, if noise  $\bar{\omega}$  satisfies the following condition

$$(C_4) : E[\bar{\omega}(\tau)] = \sum_{i=0}^{n-1} v_i \tau^i \text{ with } n \in \mathbb{N} \text{ and } v_i \in \mathbb{R},$$

and  $q \equiv w_{\mu+n, \kappa+n}^{(n)}$  with  $\mu, \kappa \in ]-\frac{1}{2}, +\infty[$ , then we have

$$\lim_{m \rightarrow +\infty} E[e_q^{\bar{\omega},m}] = 0, \quad (25)$$

and

$$e_q^{\bar{\omega},m} \xrightarrow{\mathcal{L}^2([0,1])} 0, \text{ when } T_s \rightarrow 0. \quad (26)$$

**Proof.** Recall that  $E[(Y_m - c)^2] = \text{Var}[Y_m] + (E[Y_m] - c)^2$  for any sequence of random variables  $Y_m$  with  $c \in \mathbb{R}$ , then by using Lemma 2,  $e_q^{\omega, m}$  converges in mean square to  $\int_0^1 q(\tau) E[\omega(T\tau)] d\tau$  when  $T_s \rightarrow 0$ . If  $E[\omega(\tau)] = \sum_{i=0}^{n-1} v_i \tau^i$  and  $\mu, \kappa \in ]-\frac{1}{2}, +\infty[$ , then by using the Rodrigues formula given by (1) we obtain  $w_{\mu+n, \kappa+n}^{(n)} \in \mathcal{L}^2([0, 1])$  and  $\int_0^1 w_{\mu+n, \kappa+n}^{(n)}(\tau) E[\omega(T\tau)] d\tau = 0$ . Hence, this proof is completed.  $\square$

## VI. NUMERICAL IMPLEMENTATIONS

In our identification procedure, we use a moving integration window. Hence, the estimate of  $\omega$  at  $t_i$  is given by Proposition 3 as follows

$$\forall t_i \in \Omega, \quad \tilde{\omega}^2(t_i) = -\frac{B_{y_{t_i}}}{2A_{y_{t_i}}} + \frac{\Delta_{y_{t_i}}}{2A_{y_{t_i}}}, \quad i = 0, 1, \dots, \quad (27)$$

where  $\Delta_{y_{t_i}} = \sqrt{B_{y_{t_i}}^2 - 4A_{y_{t_i}}C_{y_{t_i}}}$ ,  $A_{y_{t_i}} = T^4 I_f^{y_{t_i}, m}$ ,  $B_{y_{t_i}} = 2T^2 I_f^{y_{t_i}, m}$ ,  $C_{y_{t_i}} = I_{f(4)}^{y_{t_i}, m}$  with  $y_{t_i} \equiv y(T \cdot + t_i)$ . Note that if  $A_{y_{t_i}} = 0$ , then there is a singular value in (27). If we denote by  $\theta_i = \frac{D_{y_{t_i}}}{A_{y_{t_i}}}$  where  $D_{y_{t_i}} = -B_{y_{t_i}}$  or  $D_{y_{t_i}} = \Delta_{y_{t_i}}$ , then we can apply the following criterion (see [15]) to improve the estimation of  $\omega$

$$\min_{\theta_i \in \mathbb{R}} J(\theta_i) = \frac{1}{2} \sum_{j=0}^i v^{i+1-j} (D_{y_{t_i}} + A_{y_{t_i}} \theta_i)^2, \quad (28)$$

where  $i = 0, 1, \dots$ , and  $v \in ]0, 1]$ . The parameter  $v$  represents a forgetting factor to exponentially discard the “old” data in the recursive schema. The value of  $\theta_i$ , which minimizes the criterion (28), is obtained by seeking the value which cancels  $\frac{\partial J(\theta_i)}{\partial \theta_i}$ . Thus, we get

$$\theta_i = -\frac{\sum_{j=0}^i v^{i+1-j} D_{y_{t_i}} A_{y_{t_i}}}{\sum_{j=0}^i v^{i+1-j} (A_{y_{t_i}})^2}. \quad (29)$$

Similarly to [15], we can get the following recursive algorithm for (29)

$$\theta_{i+1} = \frac{v}{\alpha_{i+1}} (\alpha_i \theta_i + D_{y_{t_{i+1}}} A_{y_{t_{i+1}}}), \quad i = 0, 1, \dots, \quad (30)$$

where  $\alpha_i = \sum_{j=0}^i v^{i+1-j} (A_{y_{t_i}})^2$ . Moreover,  $\alpha_{i+1}$  can be recursively calculated as follows  $\alpha_{i+1} = v (\alpha_i + (A_{y_{t_i}})^2)$ .

*Example 1:* According to Section V, we can reduce the noise error part in our estimations by decreasing the sampling period. Hence, let  $(y(t_i) = x(t_i) + c\omega(t_i))_{i \geq 0}$  be a generated noise data set with a small sampling period  $T_s = 5\pi \times 10^{-4}$  in the interval  $[0, 3\pi]$  (see Fig. 1) where

$$x(t_i) = \begin{cases} \sin(10t_i + \frac{\pi}{4}), & \text{if } 0 \leq t_i \leq \pi, \\ \frac{t_i}{\pi} \sin(10t_i + \frac{\pi}{4}), & \text{if } \pi < t_i \leq 2\pi, \\ 2 \sin(10t_i + \frac{\pi}{4}), & \text{if } 2\pi < t_i \leq 3\pi, \end{cases} \quad (31)$$

and noise  $c\omega(x_i)$  is simulated from a zero-mean white Gaussian *iid* sequence with  $c = 0.1$ . Hence, the signal-to-noise ratio  $SNR = 10 \log_{10} \left( \frac{\sum |y(t_i)|^2}{\sum |c\omega(t_i)|^2} \right)$  is equal to  $SNR = 20.8\text{dB}$ . In order to estimate the frequency, by applying the previous recursive algorithm we use Proposition 1 with  $\kappa = \mu = 0$ ,  $m = 450$  and  $v = 1$ . The relating estimation error is shown in Fig. 2. By using the estimated frequency value, we estimate the amplitude and phase of the signal by applying Proposition 2 with  $\mu = 0$ ,  $m = 500$  and Proposition 4 with  $m = 500$ ,  $f_1 \equiv w_{3,2}$ ,  $f_2 \equiv w_{2,3}$ ,  $f_3 \equiv w_{3,4}$  and  $f_4 \equiv w_{4,3}$ . The relating estimation errors are shown in Fig. 3 and Fig. 4. We can observe that with small value of  $T_s$  the relating estimation errors are also small.

*Example 2:* In this example, we increase the value of  $T_s$  to  $T_s = 2\pi \times 10^{-2}$  and reduce the noise level to  $c = 0.01$ . Moreover, we add a bias term perturbation  $\xi = 0.25$  in (31) when  $t_i \in ]2\pi, 3\pi]$ . The estimations of  $\omega$  are obtained by Proposition 1 with  $\kappa = \mu = 0$ ,  $m = 12$  and  $v = 1$ . The estimations of the amplitude and phase are given by applying Proposition 2 with  $\mu = 0$ ,  $m = 12$  and Proposition 4 with  $m = 15$ ,  $f_1 \equiv w_{3,2}^{(1)}$ ,  $f_2 \equiv w_{2,3}^{(1)}$ ,  $f_3 \equiv w_{3,4}^{(1)}$  and  $f_4 \equiv w_{4,3}^{(1)}$ . The relating estimation errors are shown in Fig. 5 and Fig. 6. We can observe that the estimators obtained by modulating functions method are more robust to the sampling period and to the non zero-mean noise than the ones obtained by algebraic parametric techniques.

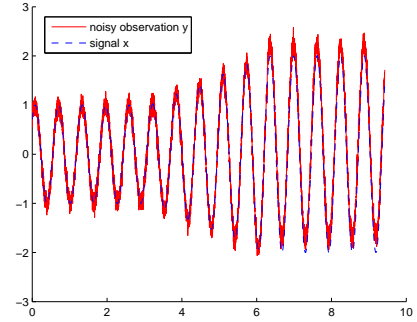


Fig. 1. The noisy observation  $y$  and the signal  $x$

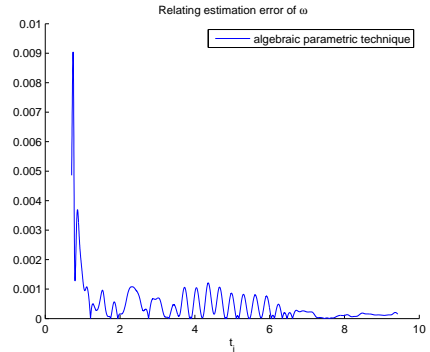


Fig. 2. Relating estimation error of  $\omega$

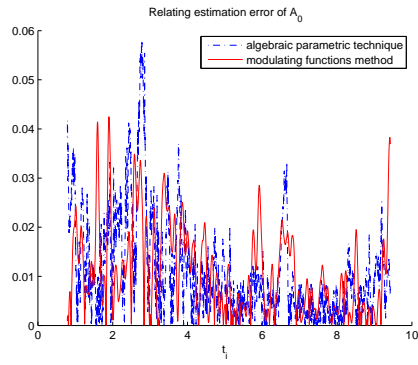


Fig. 3. Relating estimation errors of  $A_0$

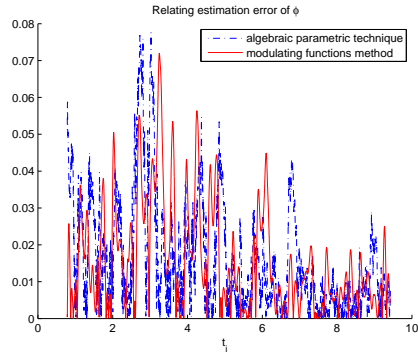


Fig. 4. Relating estimation errors of  $\phi$

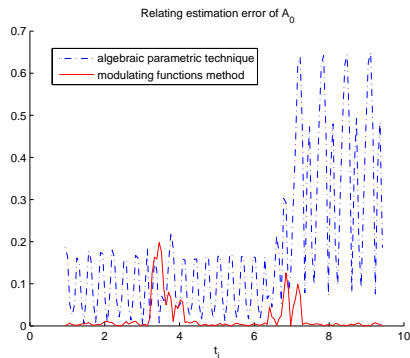


Fig. 5. Relating estimation errors of  $A_0$

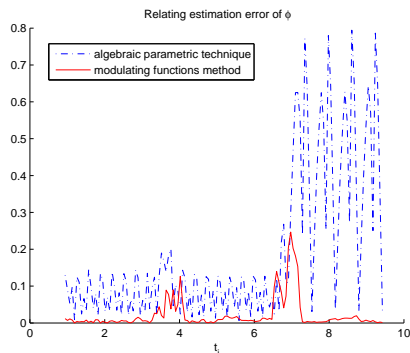


Fig. 6. Relating estimation errors of  $\phi$

## VII. CONCLUSIONS AND FUTURE WORKS

In this paper, two methods are given to estimate the frequency, amplitude and phase of a noisy sinusoidal signal with time-varying amplitude, where the estimates are obtained by using integrals. There are two types of errors for these estimates: the numerical error and the noise error part. Then, the convergence in mean square of the noise error part is studied. A recursive algorithm for frequency estimator is given. In numerical examples, we show some comparisons between the two proposed methods. Moreover, these methods can also be used to estimate the frequencies, the amplitudes and the phases of two sinusoidal signals from their noisy sum (see [11]). The analysis for colored noises will be done in a future work.

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